Logarithmic Conformal Field Theory Through Nilpotent Conformal Dimensions

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Abstract

We study logarithmic conformal field theories (LCFTs) through the introduction of nilpotent conformal weights. Using this device, we derive the properties of LCFT's such as the transformation laws, singular vectors and the structure of correlation functions. We discuss the emergence of an extra energy momentum tensor, which is the logarithmic partner of the energy momentum tensor.

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1 Introduction

Since the paper by Belavin, Polyakov and Zamolodchikov [1] on the determining role of conformal invariance on the structure of two dimensional quantum field theories, an enormous amount of work has been done on the role of conformal field theories (CFTs) in various aspects of physics such as string theory, critical phenomena and condensed matter physics. Recently Gurarie [2] has pointed to the existence of LCFTs. Correlation functions in an LCFT may have logarithmic as well as power dependence. Such logarithmic terms were ruled out earlier due to requirements such as unitarity or non existence of null states.

In an LCFT, degenerate operators exist which form a Jordan cell under conformal transformations. In the simplest case one has a pair ϕ and ψ transforming as:

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$$\begin{aligned}\phi(\lambda z) &= \lambda^{-\Delta}\phi(z),\\ \psi(\lambda z) &= \lambda^{-\Delta}[\psi(z)-\phi(z)\ln\lambda].\end{aligned} \tag{1}$$

To our knowledge the first time that logarithmic terms appear was in [3]. The emergence of the logarithmic term leads to a host of unusual properties. For instance the two point correlation functions can be derived from requirements of invariance under the action of Virasoro algebra generators $L_0, L_{\pm 1}$ [4].

$$\begin{aligned}\langle\phi(z')\phi(z)\rangle &= 0,\\ \langle\psi(z')\psi(z)\rangle &= \frac{1}{(z'-z)^{2\Delta}}(-2a\ln(z'-z)+b),\\ \langle\psi(z')\phi(z)\rangle &= \frac{a}{(z'-z)^{2\Delta}}.\end{aligned} \tag{2}$$

Generalization to higher dimensions and Jordan cells of a larger number of degenerate fields were immediate to follow [5, 6]. The development of LCFTs has had two concerns, its applications and structural matters. A good survey of both aspects can be found in [7, 8]. On the structural front, the development has been patchy. Where exactly do LCFTs fit into the larger family of CFTs requires elucidation. Interesting work on this issues has been done by Flohr [9] showing that in the minimal series, if p and q take values which are not coprime and fall out of the allowed range, LCFTs result. Also, singular vectors in LCFTs were analyzed by Flohr [10], using grassman variables, which suggests that there is a connection with supersymmetry. Connection with supersymmetry has been discussed elsewhere too [11]. More recently in papers by Gurarie and Ludwig [12] and by Kausch [13] a more direct connection with supersymmetry has been pointed out. Another exciting development is the connection between AdS theories and LCFT [14]. Indeed, if LCFTs are the natural framework for the AdS/CFT correspondence [15], then the natural supersymmetry of this theory also suggests that there is a possible connection between supersymmetry and LCFT. In this paper we follow up Flohr's [10] idea by considering a nilpotent variable θ, which is not anticommutative:

$$\begin{aligned}\theta_i^2 &= 0,\\ \theta_i\theta_j &= \theta_j\theta_i.\end{aligned} \tag{3}$$

This is perhaps equivalent to restricting ourselves to the bosonic section of a supersymmetric theory. We derive the logarithmic conformal transformations by assuming that conformal dimensions have a nilpotent part. In fact let us postulate that a primary field $\Phi(z,\theta)$ exists with the following transformation law under scaling:

$$\Phi(\lambda z,\theta) = \lambda^{-(\Delta+\theta)}\Phi(z,\theta). \tag{4}$$

However the nilpotency of θ allows us to expand both sides of equation (4). Writing $\Phi(z,\theta) = \phi(z) + \theta\psi(z)$ and $\lambda^{-(\Delta+\theta)} = \lambda^{-\Delta}(1-\theta\ln\lambda)$ equation (4) reduces to the transformation laws given in equation (1). This gives a Jordan cell of rank 2. In order to get a higher order cell one has to assume a higher level of nilpotency for θ. In general if the nth power of θ vanishes we have the following expansion for a field $\Phi(z,\theta)$

$$\Phi(z,\theta) \quad = \quad \phi_0(z) + \phi_1(z)\theta + \phi_2(z)\theta^2 + \ldots + \phi_{n-1}(z)\theta^{n-1}. \tag{5}$$

The components $(\phi_0 \ldots \phi_{n-1})$ constitute a Jordan cell of rank n. To see this it is enough to expand both sides of equation (4), using equation (5). Any how, as all the known LCFTs only contain Jordan cells of rank two, we restrict our derivations to LCFTs of this type. So, through out this paper we shall assume $\theta_i^2 = 0$. Generalization to higher rank Jordan cells is straight forward.

This paper is organized as follows. In section two we derive two and three point correlation functions. Before deriving four point functions it is necessary to discuss singular vectors (section 3) in the light of this new construction. In this section character formula are also derived. In section four we discuss the structure of the Kac determinant. In section five we derive four point functions. This leads to generalization of the hypergeometric function. The details of this calculation is discussed in appendix B. In section six we discuss the emergence of an extra energy momentum operator which is partner to the original energy momentum tensor. It appears that a pseudo identity operator and a second energy momentum tensor arise in every LCFT [2, 12]. We close this paper by discussing modifications which arise near a boundary.

2 Two and three point Correlation functions

Using transformation law of equation (4) we attempt to derive the form of correlation functions. Consider the two point functions

$$G(z_1,z_2,\theta_1,\theta_2) \quad = \quad \langle \Phi_1(z_1,\theta_1)\Phi_2(z_2,\theta_2)\rangle\,. \tag{6}$$

Within this correlation function, there exist the correlators $\langle\phi_1(z_1)\phi_2(z_2)\rangle$, $\langle\psi_1(z_1)\phi_2(z_2)\rangle$, $\langle\phi_1(z_1)\psi_2(z_2)\rangle$ and $\langle\psi_1(z_1)\psi_2(z_2)\rangle$, each of them being coefficient of 1, θ_1, θ_2 and $\theta_1\theta_2$ respectively. So deriving the correlation function (6) and expanding it in powers of θ_1 and θ_2, one easily finds the two point functions of the fields in the theory.

Translation invariance requires G, to be a function of $z_1 - z_2$ only. For rotation and scale invariance we must have

$$G(\lambda(z_1-z_2),\theta_1,\theta_2) \quad = \quad \lambda^{-(\Delta_1+\theta_1)}\lambda^{-(\Delta_2+\theta_2)}G((z_1-z_2),\theta_1,\theta_2), \tag{7}$$

Choosing $\lambda = (z_1-z_2)^{-1}$, we get

$$G(z_1-z_2,\theta_1,\theta_2) \quad = \quad \frac{1}{(z_1-z_2)^{\Delta_1+\Delta_2+\theta_1+\theta_2}}f(\theta_1,\theta_2). \tag{8}$$

For symmetry under special conformal transformations

$$z \longrightarrow \frac{1}{z+b}, \tag{9}$$

firstly we observe that equation (8) is inconsistent unless $\Delta_1 = \Delta_2$ or $f = 0$. Setting $\Delta_1 = \Delta_2$ we then get:

$$f(\theta_1, \theta_2) = f(\theta_1, \theta_2)(z_1 + b)^{(\theta_1 - \theta_2)}(z_2 + b)^{(\theta_2 - \theta_1)}. \tag{10}$$

This does not lead to a vanishing f, since θ_i's are nilpotent. Specializing to the case of a rank 2 Jordan cell, we have as the most general possible solution of equation (10):

$$f(\theta_1, \theta_2) = a_1(\theta_1 + \theta_2) + a_{12}\theta_1\theta_2. \tag{11}$$

This leads to the two point function

$$\langle \Phi(z_1, \theta_1)\Phi(z_2, \theta_2) \rangle = \frac{1}{(z_1 - z_2)^{2\Delta + (\theta_1 + \theta_2)}}(a_1(\theta_1 + \theta_2) + a_{12}\theta_1\theta_2) \tag{12}$$

which is identical to equation (2), if we expand $\Phi(z_i, \theta_i)$. The above results are easily generalized to a Jordan cell of rank n, and also to arbitrary dimensions.

Let us next consider the three point function

$$G(z_1, z_2, z_3, \theta_1, \theta_2, \theta_3) = \langle \Phi_1(z_1, \theta_1)\Phi_2(z_2, \theta_2)\Phi_3(z_3, \theta_3) \rangle . \tag{13}$$

Again it is clear that if one obtains this correlation function, all the correlators such as $\langle \phi_1\phi_2\phi_3 \rangle$, $\langle \phi_1\phi_2\psi_3 \rangle$, $\cdots$ can be calculated readily by expanding this correlation function in terms of θ_1, θ_2 and θ_3. Also note that these fields may belong to different Jordan cells. The procedure of finding this correlator is just the same as the one we did for the two point function. Like ordinary CFTs, the three point function is obtained up to some constants. Of course, in our case it is found up to a function of θ_i's, *i.e.*

$$G(z_1, z_2, z_3, \theta_1, \theta_2, \theta_3) = f(\theta_1, \theta_2, \theta_3) z_{12}^{-a} z_{23}^{-b} z_{31}^{-c}, \tag{14}$$

where $z_{ij} = (z_i - z_j)$ and

$$\begin{aligned} a &= \Delta_1 + \Delta_2 - \Delta_3 + (\theta_1 + \theta_2 - \theta_3), \\ b &= \Delta_2 + \Delta_3 - \Delta_1 + (\theta_2 + \theta_3 - \theta_1), \\ c &= \Delta_3 + \Delta_1 - \Delta_2 + (\theta_3 + \theta_1 - \theta_2). \end{aligned} \tag{15}$$

There are some constraints on $f(\theta_1, \theta_2, \theta_3)$, but further reduction requires specification the rank of Jordan cell. As we have taken it to be 2, we have:

$$f(\theta_1, \theta_2, \theta_3) = \sum_{i \neq j \neq k}^{3} C_i(\theta_j + \theta_k) + \sum_{1 \leq i < j \leq 3} C_{ij}\theta_i\theta_j + C_{123}\theta_1\theta_2\theta_3. \tag{16}$$

While symmetry considerations do not rule out a constant term on the right hand side of equation (16), but a consistent OPE forces this constant to vanish [16]. This form together with the equations (14) - (16) leads to correlation functions already obtained in the literature. This is also consistent with the observation that in all the known LCFT's so far the three point function of the first field in the Jordan cell vanishes.

In a similar fashion one can derive the form of the four point functions. But before this is done, we need to address the question of singular vectors in an LCFT.

3 Singular Vectors in LCFT

Considering the infinitesimal transformation consistent with equation (4) we have :

$$\delta\Phi \quad = \quad -\varepsilon(z^{n+1}\frac{\partial}{\partial z} + (n+1)(\Delta + \theta)z^n)\Phi. \tag{17}$$

This defines the action of the generators of the Virasoro algebra on the primary fields and points to the existence of a highest weight vector with nilpotent eigenvalue:

$$\begin{aligned} L_0|\Delta + \theta\rangle &= (\Delta + \theta)|\Delta + \theta\rangle, \\ L_n|\Delta + \theta\rangle &= 0, \qquad n \geq 1 \ . \end{aligned} \tag{18}$$

Nilpotent state $|\Delta + \theta\rangle$ can be considered as:

$$|\Delta + \theta\rangle = |\phi\rangle + \theta|\psi\rangle. \tag{19}$$

It can be easily seen that the law written in equation (19), leads to the well known equations:

$$\begin{aligned} L_0|\phi\rangle &= \Delta|\phi\rangle, \\ L_0|\psi\rangle &= \Delta|\psi\rangle + |\phi\rangle. \end{aligned} \tag{20}$$

However the norm of this vector is complex. As discussed in [18], the norm of the state $|\phi\rangle$ is zero and the states $|\phi\rangle$ and $|\psi\rangle$ are not orthogonal to each other. Instead of orthogonality condition, one can choose

$$\langle\psi|\phi\rangle = \langle\phi|\psi\rangle = 1. \tag{21}$$

The norm of $|\psi\rangle$ is not well defined and it is taken to be d. Putting all these together, one can define:

$$\langle\Delta + \theta|\Delta + \theta\rangle = \theta + \bar{\theta} + d\ \bar{\theta}\theta. \tag{22}$$

In addition to these highest weight states, there are descendants which can be obtained by applying L_{-n}'s on the highest weight vectors:

$$|\Delta + n_1 + n_2 + \cdots + n_k + \theta\rangle \quad = \quad L_{-n_1} L_{-n2} \cdots L_{-n_k} |\Delta + \theta\rangle. \tag{23}$$

To acquire more information about the content of descendants level by level, and hence the secondary operators, one usually computes the character formula:

$$\chi_\Delta(\theta, \bar{\theta}) \quad = \quad \sum_N \langle N + \Delta + \theta | \eta^{L_0 - \frac{c}{24}} | N + \Delta + \theta\rangle \tag{24}$$

which by equation (23) simplifies to

$$\chi_\Delta(\theta, \bar{\theta}) \quad = \quad \eta^{\Delta + \theta - \frac{c}{24}} \sum_N \eta^N g(N, \theta) \langle \Delta + \theta | \Delta + \theta\rangle. \tag{25}$$

Writing $g(N, \theta) = g_0(N) + \theta g_1(N)$ we obtain four characters:

$$\begin{aligned}
\chi_\Delta^{(\phi,\phi)} &= 0, \\
\chi_\Delta^{(\phi,\psi)} &= \chi_\Delta^{(\psi,\phi)} = \eta^{\Delta - \frac{c}{24}} \sum_N \eta^N g_0(N), \\
\chi_\Delta^{(\psi,\psi)} &= \eta^{\Delta - \frac{c}{24}} \sum_N \eta^N \left[g_1(N) + (d + \ln \eta) g_0(N) \right].
\end{aligned} \tag{26}$$

Appearance of logarithms in character formula have been discussed in [9, 18].

There is a submodule in which states transform among themselves, under any conformal transformation. Such a submodule is generated from a state of the form given in equation (19), such that $L_k|\chi^n_{\Delta,c}(\theta)\rangle = 0 \quad , \quad k \geq 1$. Properties of L_k imply that it is sufficient to have:

$$L_k|\chi^n_{\Delta,c}(\theta)\rangle \quad = \quad 0, \qquad k = 1, 2. \tag{27}$$

Such a vector is a linear combination of descendant vectors of level n, so we can write:

$$|\chi^n_{\Delta,c}(\theta)\rangle \quad = \quad \sum_{\{n_1+n_2+\ldots+n_m=n\}} b^{(n_1,n_2,\cdots,n_m)} L_{-n_m} \ldots L_{-n_1} |\Delta + \theta\rangle. \tag{28}$$

A singular vector must also be orthogonal to the whole vector module and in particular itself. At level n, there are $p(n)$, (partition of n) unknown coefficients in a singular vector $|\chi^n_{\Delta,c}\rangle$ which need to be determined. The action of L_1 on equation (28) provides $p(n-1)$ relations among the coefficients. One of the coefficients is arbitrary and the requirement that $L_2|\chi^n_{\Delta,c}(\theta)\rangle = 0$, gives $p(n-2)$ relations. In the following we determine the null vectors at level 2 for a rank 2 Jordan cell. We thus have:

$$|\chi^2_{\Delta,c}(\theta)\rangle \quad = \quad \left(b^{(1,1)} L^2_{-1} + b^{(2)} L_{-2} \right) |\Delta + \theta\rangle, \tag{29}$$

by applying L_1 we have:

$$\left[b^{(1,1)} [4(\Delta + \theta) + 2] L_{-1} + 3 b^{(2)} L_{-1} \right] |\Delta + \theta\rangle \quad = \quad 0. \tag{30}$$

Thus we have the solution $b^{(1,1)} = 3$ and

$$b^{(2)} \quad = \quad -[4(\Delta + \theta) + 2]. \tag{31}$$

Further more

$$[6b^{(1,1)}(\Delta + \theta) + b^{(2)}(4\Delta + \frac{c}{2} + 4\theta)]|\Delta + \theta\rangle \quad = \quad 0. \tag{32}$$

We thus get:

$$16\Delta^2 + 2\Delta(c-5) + c + 2\theta(16\Delta + c - 5) \quad = \quad 0. \tag{33}$$

We thus observe that for $\Delta = -\frac{5}{4}$ or $\frac{1}{4}$, a logarithmic null vector exists if $c = 25$ and 1 respectively. Therefore the if we write $|\chi^2_{\Delta,c}(\theta)\rangle = |\chi^2_{\Delta,c}(0)\rangle + \theta|\chi^2_{\Delta,c}(1)\rangle$ then

$$\begin{aligned} |\chi^2_{\Delta,c}(0)\rangle &= \left[3L^2_{-1} - (4\Delta + 2)L_{-2}\right]|\phi\rangle, \\ |\chi^2_{\Delta,c}(1)\rangle &= \left[3L^2_{-1} - (4\Delta + 2)L_{-2}\right]|\psi\rangle - 4L_{-2}|\phi\rangle. \end{aligned} \tag{34}$$

By the same technique logarithmic singular vectors can be obtained at higher levels. In the appendix A we present a level 3 singular vector. These results are consistent with findings of [10]. However there is evidence that, this method does not give all singular vectors [23], perhaps conditions of equation (27) are too strong.

4 Kac determinant in LCFT

In section 3 we discussed the singular vectors which occur at level 2 and 3 for a rank 2 Jordan cell. We saw that unlike ordinary CFT in an LCFT, for some special values of Δ and c singular vectors exist. Our technique in section 3 for finding singular vectors was that, we demanded at level n, $|\chi^n_{\Delta,c}(\theta)\rangle$ be a primary field in the sense that $L_1|\chi^n_{\Delta,c}(\theta)\rangle = L_2|\chi^n_{\Delta,c}(\theta)\rangle = 0$. As in an ordinary CFT we can relate the discussion of singular vectors in LCFT to Kac determinant. So it is necessary to find the form of the Kac determinant in the nilpotent θ formalism.

In ordinary CFTs, the Kac determinant arises out of setting $\langle\Delta|M|\Delta\rangle$ to zero where the elements of M have a general form like $L^m_i L^n_j \cdots$. The nonzero contributions only come from parts like L^k_0 and these terms are replaced by $\Delta^k\langle\Delta|\Delta\rangle = \Delta^k$. In our case all the above is true except that $|\Delta\rangle$ should be replaced by $|\Delta + \theta\rangle$, and since $L_0|\Delta + \theta\rangle = (\Delta + \theta)|\Delta + \theta\rangle$ all Δ's are replaced by $\Delta + \theta$. However now the norm of the state $|\Delta + \theta\rangle$ is given by equation (22), and all the coefficients of θ and $\bar{\theta}$ have to vanish independently. One thus concludes that the determinant of the matrix M should vanish. So, at level n Kac determinant has the form:

$$\det{}_n(c, \Delta + \theta) = \prod_{r,s=1;1\leq rs\leq n}^{n} (\Delta + \theta - \Delta_{r,s}(c))^{p(n-rs)}, \tag{35}$$

where $\Delta_{r,s}(c)$ is

$$\Delta_{r,s}(c) = \frac{1}{96}\left[(r+s)\sqrt{1-c} - (r-s)\sqrt{25-c}\right]^2 - \frac{1-c}{24}. \tag{36}$$

and $p(n-rs)$ is the number of partitions of the integer $n-rs$. In an ordinary CFT when the Kac determinant vanishes we have a singular vector. In our case and for a rank 2 Jordan cell the condition of vanishing $\det_n(c, \Delta+\theta)$ are:

(i) If $p(n-rs) \geq 2$ for some r and s, Kac determinant vanishes for all values of Δ that satisfy in $\Delta = \Delta_{r,s}(c)$.

(ii) If $p(n-rs) = 1$ for some pairs of $(r,s) = (r_1, s_1), (r_2, s_2), \cdots$ we can have vanishing determinant if at least $\Delta = \Delta_{r_i,s_i}(c) = \Delta_{r_j,s_j}(c)$. In this case unlike (i) we are limited to special values for Δ and c which last condition is held. As an example we consider Kac determinant at level 3. Since

$$p(3-rs) = \begin{cases} 2 & \text{r=1 , s=1} \\ 1 & \text{r=1 , s=2 or r=2 , s=1} \\ 1 & \text{r=1 , s=3 or r=3 , s=1} \end{cases}, \tag{37}$$

and $\Delta_{1,1} = 0$ and other $\Delta_{r,s}$ are nonzero, therefore $\Delta_{1,1}$ is a case of (i) and others are cases of (ii)

$$\begin{aligned} \Delta_{1,3} &= \Delta_{3,1} \Rightarrow \begin{cases} c = 1, \Delta = 1 \\ c = 25, \Delta = -3 \end{cases}, \\ \Delta_{1,2} &= \Delta_{2,1} \Rightarrow \begin{cases} c = 1, \Delta = \frac{1}{4} \\ c = 25, \Delta = \frac{-5}{4} \end{cases}, \\ \Delta_{1,3} &= \Delta_{2,1} \Rightarrow c = 28, \Delta = -2, \\ \Delta_{3,1} &= \Delta_{1,1} \Rightarrow c = -2, \Delta = 0. \end{aligned} \tag{38}$$

$$\tag{39}$$

These results are consistent with those of [10]. We observe that LCFTs are possible only when the Kac determinant has multiple zeros. This means that we have an LCFT only when degenerate conformal weights exist. Cases have been seen before, the simplest LCFT is $c = -2$ first given by [2].

5 Four point functions

To obtain further information about the theory with which we are concerned, such as surface critical exponents, OPE structure, monodromy group etc. one should compute four point correlation functions. In the language we have developed so far, the four point correlation functions depend on four θ's in addition to the coordinates of points:

$$\begin{aligned} G(z_1, z_2, z_3, z_4, \theta_1, \theta_2, \theta_3, \theta_4) &= \langle \Phi_1(z_1, \theta_1) \ldots \Phi_4(z_4, \theta_4) \rangle \\ &= f(\eta, \theta_1, \theta_2, \theta_3, \theta_4) \prod_{1 \leq i \leq j \leq 4} z_{ij}^{\mu_{ij}}. \end{aligned} \tag{40}$$

where

$$\mu_{ij} = \frac{1}{3} \sum_{k=1}^{4} (\Delta_k + \theta_k) - (\Delta_i + \theta_i) - (\Delta_j + \theta_j) \qquad , \qquad \eta = \frac{z_{41} z_{23}}{z_{43} z_{21}} \tag{41}$$

This form is invariant under all conformal transformations. Although there is no other restrictions on G due to symmetry considerations, but because of OPE structure, the four-point function $\langle\phi\phi\phi\phi\rangle$ should vanish [16, 17], that is, the term independent of θ_i's in G is zero. Thus in addition to the differential equations which should be satisfied by G, one must impose the condition $\langle\phi\phi\phi\phi\rangle = 0$ on the solution derived.

If there is a singular vector in the theory, a differential equation can be derived for $f(\eta, \theta_1, \theta_2, \theta_3, \theta_4)$. Let us consider a theory which contains a singular vector of level two. As seen in previous section the singular vector in such a theory is:

$$\chi^{(2)}(z_4, \theta_4) = \left[3L_{-1}^2 - (2(2\Delta_4 + 1) + 4\theta_4)L_{-2}\right]\Phi_4(z_4, \theta_4). \tag{42}$$

As this vector is orthogonal to all the other operators in the Verma module

$$\langle\Phi_1\Phi_2\Phi_3\chi^{(2)}\rangle \quad = \quad 0, \tag{43}$$

one immediately is led to the differential equation:

$$\left[3\partial_{z_4}^2 - (2(2\Delta_4 + 1) + 4\theta_4)\sum_{i=1}^{3}\frac{\Delta_i + \theta_i}{(z_i - z_4)^2} - \frac{\partial_{z_i}}{z_i - z_4}\right]\langle\Phi_1\Phi_2\Phi_3\Phi_4\rangle = 0. \tag{44}$$

By sending points to $z_1 = 0, \quad z_2 = 1, \quad z_3 \longrightarrow \infty$ and $z_4 = \eta$, we find:

$$\begin{aligned}\partial_\eta^2 f + [\frac{2\mu_{14}}{\eta} - \frac{2\mu_{24}}{1-\eta} - \alpha\frac{2\eta - 1}{\eta(1-\eta)}]\partial_\eta f + [\frac{\mu_{14}(\mu_{14} - 1)}{\eta^2} + \frac{\mu_{24}(\mu_{24} - 1)}{(1-\eta)^2} \\ -\frac{2\mu_{14}\mu_{24}}{\eta(1-\eta)} - \frac{\alpha(\Delta_1 + \theta_1 - \mu_{14})}{\eta^2} - \frac{\alpha(\Delta_2 + \theta_2 - \mu_{24})}{(1-\eta)^2} + \frac{\alpha\mu_{12}}{\eta(1-\eta)}]f \quad = \quad 0,\end{aligned} \tag{45}$$

where $\alpha = \frac{1}{3}\left[2(2\Delta_4 + 1) + 4\theta_4\right]$. Renormalizing using

$$H(\eta, \theta_1, \theta_2, \theta_3, \theta_4) \quad = \quad \eta^{-\beta_1 + \mu_{14}}(1-\eta)^{-\beta_2 + \mu_{24}}f(\eta, \theta_1, \theta_2, \theta_3, \theta_4), \tag{46}$$

we find that β_i satisfy

$$\beta_i(\beta_i - 1) + \alpha(\beta_i - \Delta_i - \theta_i) \quad = \quad 0, \qquad i = 1, 2 \tag{47}$$

and H satisfies the hypergeometric equation:

$$\eta(1-\eta)\frac{d^2H}{d\eta^2} + [c - (a + b + 1)\eta]\frac{dH}{d\eta} - abH \quad = \quad 0. \tag{48}$$

where

$$\begin{aligned} ab &= (\beta_1 + \beta_2)(\beta_1 + \beta_2 + 2\alpha - 1) + \alpha(\Delta_4 - \Delta_3 + \theta_4 - \theta_3), \\ a + b + 1 &= 2(\beta_1 + \beta_2 + \alpha), \\ c &= 2\beta_1 + \alpha. \end{aligned} \tag{49}$$

We can now write down the solution of equation (48) in terms of the hypergeometric series

$$H(a,b,c;\eta) = K(\theta_1,\theta_2,\theta_3,\theta_4)h(a,b,c;\eta), \tag{50}$$

where

$$h(a,b,c;\eta) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n}\eta^n, \tag{51}$$

with $(x)_n = x(x+1)\ldots(x+n-1)$, $(x)_0 = 1$. And

$$K(\theta_1,\theta_2,\theta_3,\theta_4) = \sum_{i=1}^{4} k_i\theta_i + \sum_{1\le i<j\le 4} k_{ij}\theta_i\theta_j + \sum_{1\le i<j<k\le 4} k_{ijk}\theta_i\theta_j\theta_k + k_{1234}\theta_1\theta_2\theta_3\theta_4. \tag{52}$$

Note that the coefficients in equation (51) contain nilpotent terms. Therefore equation (51) actually describes more than one solution. This expansion results in 16 functions. This is natural because as seen in the case of two and three point functions, inside the correlator (40) there exist sixteen distinct correlation functions. Of course, if all the fields belong to the same Jordan cell, only four of them may be independent and the rest are related by crossing symmetry. Also, one of them which only contains ϕ fields, vanishes and hence only three independent functions remain. The form of these functions may be obtained by expanding equation (50) and collecting powers of θ_i's. Note that expanding equation (48) leads to sixteen differential equations all of which are not independent. The general form of these equations is given in appendix B. As an example we solve equation (48) for the special case of $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = \frac{1}{4}$. In this case one of the solutions of equation (47) is:

$$\begin{aligned}
\beta_1 &= \frac{1}{2} + \theta_1 - \frac{1}{3}\theta_4 + \frac{2}{3}\theta_1\theta_4, \\
\beta_2 &= \frac{1}{2} + \theta_2 - \frac{1}{3}\theta_4 + \frac{2}{3}\theta_2\theta_4.
\end{aligned} \tag{53}$$

We then find from equation (49):

$$\begin{aligned}
a &= 2 + \theta_1 + \theta_2 + \theta_3 + \theta_4 + \frac{2}{3}(\theta_1 + \theta_2 + \theta_3)\theta_4, \\
b &= 1 + \theta_1 + \theta_2 - \theta_3 + \frac{1}{3}\theta_4 + \frac{2}{3}(\theta_1 + \theta_2 - \theta_3)\theta_4, \\
c &= 2 + 2\theta_1 + \frac{2}{3}\theta_4 + \frac{4}{3}\theta_1\theta_4.
\end{aligned} \tag{54}$$

Finally from equations (40), (46) and (50) we get expressions for the various four point functions. For example

$$\langle\psi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)\rangle = k_1\eta^{\frac{2}{3}}(1-\eta)^{\frac{2}{3}}h_0(\eta)\prod_{1\le i<j\le 4} z_{ij}^{-\frac{1}{6}}. \tag{55}$$

Where $h_0(\eta)$ is given in appendix B. The other four point functions having 2, 3 or 4 ψ's can be calculated in the same way. In these functions, depending on how many

ψ's are present in the correlators, different functions appear on the right hand side. If there is no ψ in the correlator, the correlator is zero, if there is one $\psi(z_i)$, only H_i appears, if there are two ψ's, H_i and H_{ij} appear, and so on. The situation is just the same for the two or three point functions. For example the two point function $\langle\phi(z)\psi(0)\rangle$ is written in terms of the function $z^{-2\Delta}$ and in the correlator $\langle\psi(z)\psi(0)\rangle$ there exist both functions $z^{-2\Delta}$ and $z^{-2\Delta}\ln z$.

6 Energy momentum tensor

Two central operators in a CFT are the energy momentum tensor, T with conformal weight $\Delta = 2$ and the identity operator, I with conformal weight $\Delta = 0$. However, T is a secondary field of the identity, because $L_{-2}I = T$.

In an LCFT degenerate operators exist which form a Jordan cell under conformal transformation. This holds true for the identity as well. The existence of a logarithmic identity operator has been discussed by a number of authors [2, 19, 20].

Now consider the identity operator I_0 and its logarithmic partner I_1. According to equation (1) this pair transforms as:

$$\begin{aligned} I_0(\lambda z) &= I_0(z), \\ I_1(\lambda z) &= I_1(z) - I_0(z)\ln\lambda. \end{aligned} \tag{56}$$

So according to our convention, we define a primary field $I(z,\theta)$:

$$I(z,\theta) = I_0(z) + \theta I_1(z). \tag{57}$$

with conformal weight θ. Under scaling, $I(z,\theta)$ transforms according to equation (4). Thus we have:

$$\begin{aligned} L_0 I_0(z) &= 0, \\ L_0 I_1(z) &= I_0(z). \end{aligned} \tag{58}$$

this was first observed in $c = -2$ theory by Gurarie [2].

Here we wish to find the field $T(z,\theta)$ with conformal weight $2+\theta$ which is a secondary of $I(z,\theta)$ in the sense:

$$L_{-2}I(z,\theta) = T(z,\theta). \tag{59}$$

By writing $T(z,\theta) = T_0(z) + \theta T_1(z)$ and since $L_0 T(z,\theta) = (2+\theta)T(z,\theta)$ we have:

$$\begin{aligned} L_0 T_0(z) &= 2T_0(z), \\ L_0 T_1(z) &= 2T_1(z) + T_0(z). \end{aligned} \tag{60}$$

This points to the existence of an extra energy momentum tensor [12, 19, 20]. By applying L_2 on both sides of equation (59) we have:

$$\begin{aligned} L_2 T_0(z) &= \frac{c}{2} I_0(z), \\ L_2 T_1(z) &= \frac{c}{2} I_1(z) + 4I_0(z). \end{aligned} \tag{61}$$

The first of this pair exists in an ordinary CFT, so $T_0(z)$ leads to the Virasoro algebra, while $T_1(z)$ must leads to a new algebra [12]. We now attempt at finding the OPE of the extra energy momentum tensor, $T_1(z)$ with $\Phi(z,\theta)$ and $T(z,\theta)$. Because of OPE's invariance under scaling and according to our convention it is sufficient to change conformal weight of each field to $\Delta+\theta$. Consider the following OPE:

$$T_0(z')\Phi(z,\theta) = \frac{\Delta+\theta}{(z'-z)^2}\Phi(z,\theta) + \frac{\partial_z\Phi(z,\theta)}{z'-z} + \cdots \quad . \tag{62}$$

This relation leads to the familiar OPE for $T(z')\phi(z)$ a new OPE:

$$T_0(z')\psi(z) = \frac{\phi(z)+\Delta\psi(z)}{(z'-z)^2} + \frac{\partial_z\psi(z)}{z'-z} + \cdots \quad . \tag{63}$$

Also

$$T_0(z')T(z,\theta) = \frac{\frac{c(\theta)}{2}I(\theta)}{(z'-z)^4} + \frac{2+\theta}{(z'-z)^2}T(z,\theta) + \frac{\partial_z T(z,\theta)}{z'-z} + \cdots \quad , \tag{64}$$

where $c(\theta) = c_1 + \theta c_2$. Again we obtain two OPE, one of them is $T(z')T(z)$ which is known from CFT and the other is:

$$T_0(z')T_1(z) = \frac{\frac{c_1}{2}I_1(z) + \frac{c_2}{2}I_0}{(z'-z)^4} + \frac{T(z)+2T_1(z)}{(z'-z)^2} + \frac{\partial_z T_1(z)}{z'-z} + \cdots \quad . \tag{65}$$

The emergence of an extra energy momentum tensor and central charge have been noticed by Gurarie and Ludwig [12], although our approach is very different.

It is worth noting that equations (56) imply that $\langle I_0\rangle$ vanishes whereas $\langle I_1\rangle = 1$. This immediately results in the vanishing of $\langle T_0 T_0\rangle$.

7 Boundary

Let us now consider the problem of LCFT near a boundary. As shown in [21] in an ordinary CFT, if the real axis is taken to be the boundary, with certain boundary condition that $T = \overline{T}$ on the real axis, the differential equation satisfied by n-point function near a boundary are the same as the differential equations satisfied by 2n-point function in the bulk. This trick may be used in order to derive correlations of an LCFT near a boundary [18, 22]. Here we rederive the same results using the nilpotent formalism. Again we consider an LCFT with a rank 2 Jordan cell. First we find the one point functions of this theory. By applying $L_0, L_{\pm 1}$ on the correlators, one obtains:

$$\begin{aligned} (\partial_z + \partial_{\bar z})\langle\Phi(z,\bar z,\theta)\rangle &= 0, \\ (z\partial_z + \bar z\partial_{\bar z} + 2(\Delta+\theta)\langle\Phi(z,\bar z,\theta\rangle &= 0, \\ (z^2\partial_z + \bar z^2\partial_{\bar z} + 2z(\Delta+\theta) + 2\bar z(\Delta+\theta))\langle\Phi(z,\overline{z},\theta\rangle &= 0. \end{aligned} \tag{66}$$

In these equations, we have assumed that Φ is a scalar field so that $\Delta = \bar\Delta$. The first equation states $\langle\Phi(z,\bar z,\theta)\rangle$ is a function of $z-\bar z$ and the solution to the second

equation is:

$$\langle\Phi(y,\theta)\rangle = \frac{f(\theta)}{y^{2(\Delta+\theta)}}, \tag{67}$$

where $y = z - \bar{z}$. The third line of equation (66) is automatically satisfied by this solution. Expanding $f(\theta)$ as $a + b\theta$ one finds:

$$\langle\Phi(y,\theta)\rangle = \frac{a}{y^{2\Delta}} + \frac{\theta}{y^{2\Delta}}(b - 2a\ln y). \tag{68}$$

As the field $\Phi(y,\theta)$ is decomposed to $\phi(y)+\theta\psi(y)$ one can read the one-point functions $\langle\phi(y)\rangle$ and $\langle\psi(y)\rangle$ from the equation (68):

$$\langle\phi(y)\rangle = \frac{a}{y^{2\Delta}}, \tag{69}$$

$$\langle\psi(y)\rangle = \frac{1}{y^{2\Delta}}(b - 2a\ln y). \tag{70}$$

To go further, one can investigate the two-point function $G(z_1,\bar{z}_1,z_2,\bar{z}_2,\theta_1,\theta_2) = \langle\Phi(z_1,\bar{z}_1,\theta_1)\Phi(z_2,\bar{z}_2,\theta_2)\rangle$ in the same theory. Invariance under the action of L_{-1} implies:

$$(\partial_{z_1} + \partial_{\bar{z}_1} + \partial_{z_2} + \partial_{\bar{z}_2})G = 0. \tag{71}$$

The most general solution of this equation is $G = G(y_1,y_2,x_1,x_2,\theta_1,\theta_2)$, where $y_1 = z_1 - \bar{z}_1$, $y_2 = z_2 - \bar{z}_2$, $x = x_2 - x_1$ and $x_i = z_i + \bar{z}_i$. By invariance under the action of L_0 we should have :

$$[y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} + x\frac{\partial}{\partial x} + 2(\Delta+\theta_1) + 2(\Delta+\theta_2)]G = 0, \tag{72}$$

which implies:

$$G = \frac{1}{x^{4\Delta+2\theta_1+2\theta_2}}f(\alpha_1,\alpha_2,\theta_1,\theta_2). \tag{73}$$

where $\alpha_1 = \frac{y_1}{x}$ and $\alpha_2 = \frac{y_2}{x}$. Now consider the action of L_1 on G:

$$\begin{aligned}(x_1+x_2)[y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} + x\frac{\partial}{\partial x} + 2(\Delta+\theta_1) + 2(\Delta+\theta_2)]G \\ +[xy_1\frac{\partial}{\partial y_1} - xy_2\frac{\partial}{\partial y_2} + (y_1^2 - y_2^2)\frac{\partial}{\partial x} + 2x(\theta_1 - \theta_2)]G = 0.\end{aligned} \tag{74}$$

The first bracket is zero because of equation (72). Substituting the solution (73) in equation (74) the function f satisfies:

$$\left(\alpha_1 + \frac{\alpha_1}{\alpha_1^2-\alpha_2^2}\right)\frac{\partial f}{\partial\alpha_1} + \left(\alpha_2 + \frac{\alpha_2}{\alpha_2^2-\alpha_1^2}\right)\frac{\partial f}{\partial\alpha_2} + 2\left(2\Delta+\theta_1+\theta_2+\frac{\theta_1-\theta_2}{\alpha_1^2-\alpha_2^2}\right)f = 0. \tag{75}$$

The most general solution of above is:

$$f(\alpha_1,\alpha_2) \;=\; \frac{1}{(\alpha_1\alpha_2)^{2\Delta+\theta_1+\theta_2}}\left(\frac{\alpha_2}{\alpha_1}\right)^{\theta_1-\theta_2} g\left(\frac{1+\alpha_1^2+\alpha_2^2}{\alpha_1\alpha_2},\theta_1,\theta_2\right), \tag{76}$$

where g is an arbitrary function. So the two-point function G is found up to an unknown function:

$$\langle\Phi(z_1,\bar{z}_1,\theta_1)\Phi(z_2,\bar{z}_2,\theta_2)\rangle \;=\; \frac{1}{(y_1y_2)^{2\Delta+\theta_1+\theta_2}}\left(\frac{y_1}{y_2}\right)^{\theta_1-\theta_2} h\left(\frac{x^2+y_1^2+y_2^2}{y_1y_2}\right), \tag{77}$$

which is the same as the solution obtained in [22].

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8 Appendix A: Level 3 singular vector

In this appendix we derive level 3 singular vectors. According to equation (28) the general form of a singular vector at level 3 is:

$$|\chi^3_{\Delta,c}(\theta)\rangle \;=\; (b^{(1,1,1)}L_{-1}^3+b^{(1,2)}L_{-1}L_{-2}+b^{(3)}L_{-3})|\Delta+\theta\rangle. \tag{78}$$

By applying L_1 on both sides of above equation and use of Virasoro algebra and equation (18), we have:

$$[6b^{(1,1,1)}(\Delta+1+\theta)L_{-1}^2+b^{(1,2)}(3L_{-1}^2+2(\Delta+2+\theta)L_{-2})+4b^{(3)}L_{-2}]|\Delta+\theta\rangle=0. \tag{79}$$

Because L_{-1}^2 and L_{-2} are independent we must have

$$\begin{aligned} 6(\Delta+1+\theta)b^{(1,1,1)}+3b^{(1,2)} &= 0\\ 2(\Delta+2+\theta)b^{(1,2)}+4b^{(3)} &= 0. \end{aligned} \tag{80}$$

If we choose $b^{(1,1,1)}=1$ and find $b^{(1,2)}$ and $b^{(3)}$ in equation (80) we get the singular vector at level 3 for rank 2 Jordan cell as:

$$\begin{aligned} |\chi^3_{\Delta,c}(\theta)\rangle = \{L_{-1}^3-2(\Delta+1+\theta)L_{-1}L_{-2}\\ +[(\Delta+1)(\Delta+2)+(2\Delta+3)\theta]L_{-3}\}|\Delta+\theta\rangle. \end{aligned} \tag{81}$$

Now if we demand that $L_2|\chi^3_{\Delta,c}(\theta)\rangle=0$ we have:

$$[-3\Delta^2+7\Delta-2-c(\Delta+1)]+\theta[-6\Delta-c+7] \;=\; 0 \tag{82}$$

and therefore Δ and c are restricted to the values:

$$c \;=\; \frac{3\Delta^2-7\Delta+2}{\Delta+1} = -6\Delta+7. \tag{83}$$

From these relations we observe that $\Delta=-3$ or 1 which corresponds to $c=25$ and 1 respectively. For these values, the singular vector is orthogonal to any other vector. In particular, $\langle\chi^3_{\Delta,c}(\theta)|\chi^3_{\Delta,c}(\theta)\rangle=0$.

Now we are ready to find the explicit form of the singular vectors. By using equation (81) and

$$|\chi^3_{\Delta,c}(\theta)\rangle = |\chi^3_{\Delta,c}(0)\rangle + \theta|\chi^3_{\Delta,c}(1)\rangle, \tag{84}$$

we have:

$$|\chi^3_{\Delta,c}(0)\rangle \;=\; [L^3_{-1}-2(\Delta+1)L_{-1}L_{-2}+(\Delta+1)(\Delta+2)L_{-3}]|\phi\rangle \tag{85}$$

and

$$\begin{aligned}|\chi^3_{\Delta,c}(1)\rangle = [L^3_{-1}-2(\Delta+1)L_{-1}L_{-2}+(\Delta+1)(\Delta+2)L_{-3}]|\psi\rangle \\ +[-2L_{-1}L_{-2}+(2\Delta+3)L_{-3}]|\phi\rangle.\end{aligned} \tag{86}$$

As expected equation (85) is exactly the level 3 singular vector of a normal CFT. But in such theories there is no restriction on the values of c or Δ. On the other hand in LCFT's the presence of a second null vector forces us to allow only certain values of c and Δ as given by equation (83). We observe that some null vectors obtained in [10] are missing, in other words we have an incomplete set. The reason for this may be that equation (27) sets too strong a condition within LCFT [23].

9 Appendix B: Hypergeometric Functions

In this appendix we show that equation (48) can be considered as 16 differential equations. However one of them is trivial because it vanishes due to OPE constraints [16, 17]. According to equation (49) a, b and c, H are functions of θ_i's. We write any of them in a general form:

$$H=\sum_{i=1}^{4}H_i\theta_i+\sum_{1\le i<j\le 4}H_{ij}\theta_i\theta_j+\sum_{1\le i<j<k\le 4}H_{ijk}\theta_i\theta_j\theta_k+H_{1234}\theta_1\theta_2\theta_3\theta_4 \tag{87}$$

and in a similar way for a, b and c. Now by substitution of them in equation (48) we obtain 15 differential equations:

$$DH_i=0 \tag{88}$$

$$DH_{ij}=\{-[c_i-(a_i+b_i)\eta]\frac{dH_j}{d\eta}+(a_0b_i+a_ib_0)H_j+i\longleftrightarrow j\} \tag{89}$$

$$DH_{ijk} = [-[c_k - (a_k + b_k)\eta]\frac{dH_{ij}}{d\eta} + (a_0 b_k + a_k b_0)H_{ij}$$
$$-[c_{ij} - (a_{ij} + b_{ij})\eta]\frac{dH_k}{d\eta} + (a_0 b_{ij} + a_i b_j + a_{ij} b_0)H_k + cyclic \ \ terms] \tag{90}$$

$$\begin{aligned} DH_{1234} = [&-[c_l - (a_l + b_l)\eta]\frac{dH_{ijk}}{d\eta} + (a_0 b_l + a_l b_0)H_{ijk} \\ &-[c_{ij} - (a_{ij} + b_{ij})\eta]\frac{dH_{kl}}{d\eta} + (a_0 b_{ij} + a_i b_j + a_{ij} b_0)H_{kl} \\ &-[c_{ijk} - (a_{ijk} + b_{ijk})\eta]\frac{dH_l}{d\eta} + (a_0 b_{ijk} + a_{ijk} b_0 + a_k b_{ij} + a_{ij} b_k)H_l \\ &+cyclic \ \ terms] \end{aligned} \tag{91}$$

where

$$D := \eta(1-\eta)\frac{d^2}{d\eta^2} + [c_0 - (a_0 + b_0 + 1)\eta]\frac{d}{d\eta} - a_0 b_0. \tag{92}$$

Let us now obtain from equation (51), first few terms of 16 functions that are solutions of differential equations, given above for the special case of $\Delta_i = \frac{1}{4}$

$$\begin{aligned}
h_0 &= F(2,1,2,\eta) = 1 + \eta + \eta^2 + \eta^3 + \cdots \\
h_1 &= \frac{1}{2}\eta + \frac{2}{3}\eta^2 + \frac{3}{4}\eta^3 + \cdots \ , \qquad h_2 = \frac{3}{2}\eta + \frac{7}{3}\eta^2 + \frac{35}{12}\eta^3 + \cdots \\
h_3 &= -\frac{1}{2}\eta - \frac{2}{3}\eta^2 - \frac{3}{4}\eta^3 + \cdots \ , \qquad h_4 = \frac{1}{2}\eta + \frac{7}{9}\eta^2 + \frac{35}{36}\eta^3 + \cdots \\
h_{12} &= -\frac{1}{2}\eta - \frac{1}{18}\eta^2 + \frac{29}{72}\eta^3 + \cdots \ , \qquad h_{13} = \frac{1}{2}\eta + \frac{4}{9}\eta^2 + \frac{3}{8}\eta^3 + \cdots \\
h_{23} &= -\frac{2}{3}\eta^2 - \frac{5}{4}\eta^3 + \cdots \ , \qquad h_{14} = \frac{1}{3}\eta + \frac{2}{3}\eta^2 + \frac{11}{12}\eta^3 \\
h_{24} &= \frac{7}{6}\eta + \frac{70}{27}\eta^2 + \frac{281}{72}\eta^3 + \cdots \ , \quad h_{34} = -\frac{1}{2}\eta - \frac{49}{54}\eta^2 - \frac{259}{216}\eta^3 + \cdots \\
h_{123} &= \frac{7}{9}\eta^2 + \frac{7}{6}\eta^3 + \cdots \ , \qquad h_{124} = -\frac{2}{3}\eta - \frac{41}{162}\eta^2 + \frac{317}{648}\eta^3 + \cdots \\
h_{134} &= \frac{2}{3}\eta + \frac{121}{162}\eta^2 + \frac{455}{648}\eta^3 + \cdots \ , \quad h_{234} = -\frac{32}{27}\eta^2 - \frac{46}{18}\eta^3 + \cdots \\
h_{1234} &= \frac{137}{81}\eta^2 + \frac{17}{6}\eta^3 + \cdots .
\end{aligned} \tag{93}$$

where $Dh_0 = 0$.

References

[1] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. **B241** (1984) 333

[2] V.Gurarie, Nucl. Phys. **B410** (1993) 535 [hep-th/9303160]

[3] L. Rozansky, H. Saleur, Nucl. Phys. **B376** (1992) 461; Nucl. Phys. **B389** (1993) 365 [hep-th/9203069]

[4] J.-S. Caux, I. I. Kogan and A. M. Tsevelik, Nucl. Phys. **B466** (1996) 444 [hep-th/9511134]

[5] A. M. Ghezelbash, V. Karimipour, Phys. Lett. **B402** (1997) 2

[6] M. R. Rahimi-Tabar, A. Aghamohammadi and M. Khorrami, Nucl. Phys. **B497** (1997) 555 [hep-th/9610168]

[7] J. Cardy, [cond-mat/9911024]

[8] M. R. Rahimi Tabar, [cond-mat/0002309]

[9] M. Flohr, Int. J. Mod. Phys. **A11** (1996) 4147 [hep-th/9509166]; Int. J. Mod. Phys. **A12** (1997) 1943 [hep-th/9605151]

[10] M. Flohr, Nucl. Phys. **B514** (1998) 523 [hep-th/9707090]

[11] Z. Maassarani and D. Serban, Nucl. Phys. **B489** (1997) 603

[12] V. Gurarie and A. W. W. Ludwig [cond-mat/9911392]

[13] H. G. Kausch [hep-th/0003029]

[14] I. Kogan, phys. Lett **B458** (1999) 66 [hep-th/9903162]

[15] J. M. Maldacena, Adv. Theor. Math. Phys. **2** (1998) 231

[16] S. Moghimi-Araghi, S. Rouhani and M. Sadaat, [hep-th/0012149]

[17] M. Flohr, [hep-th/0009137]

[18] I. I. Kogan and J. F. Wheater [hep-th/0003184]

[19] I. I. Kogan and A. Lewis, Nucl. Phys. **B 509** (1998) 687 [hep-th/9705240]

[20] J.-S. Caux, I. I. Kogan, A. Lewis and A. M. Tsevelik, Nucl. Phys. **B489** (1997) 469 [hep-th/9606138]

[21] J. Cardy, Nucl. Phys. **B240** [FS12] (1984) 514

[22] S. Moghimi-Araghi and S. Rouhani, Lett. Math. Phys. **53** (2000) 49 [hep-th/0002142]

[23] M. Flohr, Private communication.